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Supersymmetric moduli of the $\mathrm{SU}(2) \times \mathbb{R}_\phi$ linear dilaton background and NS5-branes

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Abstract: We study several classes of marginal deformations of the conformal field theory $\mathrm{SU}(2)_k \times \mathbb{R}_\phi$. This theory describes the near-horizon region of a stack of parallel and coincident NS5-branes and is related holographically to little string theory. We investigate the supersymmetry properties of these deformations and we elucidate their rôle in the context of holography. The conformal field theory moduli space contains "non-holographic" operators that do not seem to have a simple interpretation in little string theory. Subsequently, we analyze several NS5-brane configurations in terms of $\mathrm{SU}(2)_k \times \mathbb{R}_\phi$ deformations. We discuss in detail interesting phenomena, like the excision of the strong coupling region associated with the linear dilaton and the manifestation of the symmetries of an NS5-brane setup in the deforming operators. Finally, we present a class of conformally hyperkähler geometries that arise as "non-holographic" deformations of $\mathrm{SU}(2)_k \times \mathbb{R}_\phi$.

Keywords: p-branes, Conformal Field Models in String Theory, Gauge-gravity correspondence, AdS-CFT Correspondence.

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1. Introduction

String theory backgrounds that admit an exact conformal field theory (CFT) description are of particular interest since their physical properties can be analyzed to all orders in α'. The situation is even more interesting when these backgrounds are created by the backreaction of a configuration of branes. In this case, deformations of the CFT correspond to deformations of the brane system. It often happens that some of the latter can be visualized as changes in the geometry of the branes, thereby leading to a very intuitive geometrical picture of the CFT moduli space.

Unfortunately, the number of brane systems that admit an exact CFT description is rather small. First of all, configurations with D-branes source Ramond-Ramond fields and, as is well-known, it is notoriously difficult to obtain a useful CFT description of such backgrounds. However, even when solely NS5-branes are present, there are so far only two instances where the underlying CFT is known explicitly. The first is a configuration of k parallel NS5-branes put at the same point in their transverse $\mathbb{R}^4$ space. The near-horizon region of this system is described by the Callan-Harvey-Strominger (CHS) theory $\mathrm{SU}(2)_k \times \mathbb{R}_\phi$ [1]. We will provide a brief reminder on this theory in the next section. The second instance is that of k parallel NS5-branes put uniformly on a circle in $\mathbb{R}^4$. In this case, their near-horizon region is described by the coset CFT $\mathrm{SU}(2)_k/\mathrm{U}(1) \times \mathrm{SL}(2)_k/\mathrm{U}(1)$ orbifolded by $\mathbb{Z}_k$ [2]. Both of these theories support the $\mathcal{N} = 4$ superconformal algebra, since the associated NS5-brane systems are 1/2 BPS (they preserve 16 supercharges in type II string theories and 8 supercharges in the heterotic string).

Geometric deformations of the system of NS5-branes away from the point or the circle distribution are associated with exactly marginal operators in the underlying CFT. The reason is that moving the NS5-branes away from their original locations yields a configuration that is also a solution of the equations of motion, continuously connected to the original one, and the space of such solutions is generically identified with the space of exactly marginal deformations of the CFT. Furthermore, since an arbitrary configuration of NS5-branes in $\mathbb{R}^4$ preserves the same amount of supersymmetry as the point-like configuration, we are naturally lead to consider only the deformations of the CHS theory that preserve all of the original $\mathcal{N} = 4$ worldsheet supersymmetry.

The connection between deformations of the NS5-brane system and marginal operators in the CFT can be established either directly or through the use of holography. The first approach is based on the fact that changes of the original locations of the NS5-branes correspond to perturbations of the original supergravity background that subsequently induce deformations of the associated worldsheet σ model. One can read these σ model deformations and express them in terms of operators of the original undeformed theory. The last step is performed by employing the semiclassical expression of these operators in terms of σ model target space fields. This approach to NS5-brane deformations was initiated in [3], where the operators in $\mathrm{SU}(2)_k/\mathrm{U}(1) \times \mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ that trigger an elliptical perturbation of the circular NS5-brane system were uncovered.

The second approach is based on the fact that the decoupled worldvolume theory on the NS5-branes, known as little string theory (LST), admits a holographic description in terms of string theory on the near-horizon limit of the background generated by the NS5-branes [4]. The conjectured holography implies a correspondence between operators in LST and vertex operators in the dual string theory background [5, 6]. Since the moduli space of the geometric NS5-brane deformations is the moduli space of LST and the latter is parametrized by gauge invariant LST operators, we see that one can associate in this way deformations of the NS5-branes with operators in the underlying CFT. At the level of holography this association is done by using the symmetries of the two sides of the correspondence. However, symmetry matchings do not constitute a proof and one would like to substantiate the holographic dictionary between operators in a more explicit way. This was achieved in [7] where the first approach, based on the σ model description of the deformed NS5-brane background, was used to validate the holographic correspondence in the semiclassical limit of large k.

In this paper we investigate some generic issues pertaining to deformations of the CHS theory and in conjunction with its NS5-brane interpretation. First we will perform a study of the supersymmetry properties of several types of marginal operators of the CHS. We will be particularly interested in operators that preserve the original $\mathcal{N} = 4$ superconformal symmetry of the CHS model, as these operators can in principle correspond to geometric deformations of the NS5-branes. Surprisingly, we will also uncover some other classes of supersymmetry preserving marginal deformations that do not seem to have holographic counterparts. Although we will elucidate the physical effects of some of them with some simple examples, presented in the last section, the precise understanding of their interpretation in terms of NS5-branes and LST is left for future work. Subsequently, we will consider a set of NS5-brane configurations that arise as deformations of the point-like setup and, therefore, can be described by marginal operators in the CHS theory. Our objective would be to show how the physical and geometrical properties of these configurations are encoded in the corresponding CFT operators. This analysis will illuminate further the fascinating interplay between spacetime and CFT physics.

2. Supersymmetric operators in the CHS background

In this section we study the supersymmetry properties of a class of marginal operators of the CHS background. First, we find conditions for these operators to be chiral or antichiral primaries and subsequently we check which of these operators yield supersymmetry preserving deformations. Since extended worldsheet supersymmetry is necessary for spacetime (i.e. worldvolume) supersymmetry, we check first the former and then perform a test of the latter. In the course of our analysis we will uncover that some operators that do not seem to have a holographic intepretation preserve also maximal supersymmetry.

2.1 Generalities

The holographic description of LST is based on the correspondence between BPS operators in LST and vertex operators in the CHS background $\mathrm{SU}(2)_k \times \mathbb{R}_\phi$ [5, 6]. The latter contains a linear dilaton along the ϕ direction with background charge $q = \sqrt{\frac{2}{k}}$, where k is the number of NS5-branes, and a $\mathcal{N} = 1$ supersymmetric SU(2) WZW model at level k generated by affine currents $\mathcal{J}^a, a = 1, 2, 3$. A class of BPS operators in LST consists of $\widetilde{\mathrm{tr}}(X^{i_1} X^{i_2} \cdots X^{i_{2j+2}})$ with $j = 0, \frac{1}{2}, 1, \ldots, \frac{k-2}{2}$ and where $X^i, i = 6, 7, 8, 9$ are scalar fields in the adjoint representation of $\mathrm{SU}(k)$ whose eigenvalues parametrize the transverse positions of the NS5-branes. In order that the LST operators are in a short multiplet of spacetime supersymmetry, only the traceless and symmetric components in the indices $i_1, \ldots, i_{2j+2}$ should be kept. The tilde on the trace means that we should not consider the standard single trace but its combination with multi-traces. This subtlety, however, will not play any rôle in the considerations of this section.

The dictionary proposed in [5, 6] and tested in a non-trivial setup in [7] states the correspondence

$$\widetilde{\mathrm{tr}}(X^{i_1} X^{i_2} \cdots X^{i_{2j+2}}) \longleftrightarrow (\psi\bar{\psi}\Phi_j)_{j+1;m,\bar{m}} e^{-qa_j\phi}\,, \tag{2.1}$$

where the right-hand side is an operator in the CHS theory. The coefficient a_j of the linear dilaton vertex operator at the right must be either $a_j = j + 1$ or $a_j = -j$ in order that the actual deforming operators, which arise from the action of the $\mathcal{N}=1$ supercharges on the operators at the right-hand side of (2.1), are marginal (see formula (2.22) below). In the first case the operator is normalizable[1] and hence it corresponds to a situation where the dual LST operator acquires a vacuum expectation value.

The way one associates geometric deformations of the NS5-branes to CHS operators using the above holographic correspondence is the following. The original configuration of NS5-branes put at the point $x^6 = x^7 = x^8 = x^9 = 0$ is described, in the near-horizon limit, by the unperturbed CHS theory. A generic point in the moduli space, which corresponds to separating the branes in their transverse $\mathbb{R}^4$, thereby turning on non-vanishing expectation values for the scalars X^i, is described by a deformation of the original CFT with operators that can be found using the correspondence (2.1) (according to formula (2.22) below). Notice that we consider only deformations that leave invariant the center of mass of the NS5-brane system, in other words we always assume that $\mathrm{tr}(X^i) = 0$. The reason is that the associated U(1) degree of freedom in LST is frozen and decouples, so that there is no normalizable mode corresponding to it. The other value of a_j that yields also a marginal deformation, i.e. $a_j = -j$, corresponds to a non-normalizable deformation of the CHS theory that triggers a perturbation of the LST with the operator at the left-hand side.

In order to write the CFT operators explicitly, we decompose the supersymmetric WZW model into a bosonic $\mathrm{SU}(2)_{k-2}$ WZW model at level $k-2$, whose affine currents we will denote by J^i, and three free fermions ψ^a, $a = 1,2,3$ in the adjoint of SU(2). Consequently, the $\mathcal{N}=1$ affine currents can be written as $\mathcal{J}^a = J^a - \frac{i}{2}\epsilon^{abc}\psi^b\psi^c$. The field Φ_j is in general a Virasoro primary of the bosonic $\mathrm{SU}(2)_{k-2}$ WZW model and the notation $(\psi\bar{\psi}\Phi_j)_{j+1;m,\bar{m}}$ means that we should tensor the fermions ψ_a to the bosonic primary Φ_j into a primary of total spin $j+1$ and $(\mathcal{J}^3, \bar{\mathcal{J}}^3) = (m, \bar{m})$.

It will be practical to introduce the complex fermiom combinations $\psi^{\pm} = \frac{1}{\sqrt{2}}(\psi_1 \pm i\psi_2)$ and also perform the usual change of basis for the $\mathrm{SU}(2)_{k-2}$ currents $J^{\pm} = J^1 \pm iJ^2$. The super-affine currents then read $\mathcal{J}^3 = J^3 + \psi^+\psi^-$ and $\mathcal{J}^{\pm} = J^{\pm} \pm \sqrt{2}\psi^3\psi^{\pm}$. Finally, we will use extensively the $\mathrm{SU}(2)_{k-2}$ current algebra at level $k-2$

$$J^3(z)J^3(w) \sim \frac{k-2}{2(z-w)^2}\,, \tag{2.2}$$

$$J^3(z)J^{\pm}(w) \sim \pm\frac{J^{\pm}(w)}{z-w}\,, \tag{2.3}$$

$$J^+(z)J^-(w) \sim \frac{k-2}{(z-w)^2} + \frac{2J^3(w)}{z-w}\,, \tag{2.4}$$

[1]Notice that strictly speaking one cannot talk meaningfully about normalizable operators in the CHS theory since they are supported in the strong coupling region $\phi \to -\infty$. However, there is a 1-1 correspondence between operators of the $\mathrm{SU}(2)_k \times \mathbb{R}_\phi$ theory and of the non-singular coset CFT $\mathrm{SL}(2)_k/\,\mathrm{U}(1) \times \mathrm{SU}(2)_k/\,\mathrm{U}(1)$, so that all of our subsequent discussion can be trivially generalized to the physically more reliable coset theory.

and the action of the $SU(2)_{k-2}$ currents on the Virasoro primaries $\Phi_{j;m}$:

$$\begin{aligned} J^3(z)\Phi_{j;m}(w) &\sim \frac{m}{z-w}\Phi_{j;m}(w)\,, \\ J^{\pm}(z)\Phi_{j;m}(w) &\sim \frac{j\mp m}{z-w}\Phi_{j;m\pm1}(w)\;. \end{aligned} \tag{2.5}$$

Now, we can write explicitly

$$(\psi\bar{\psi}\Phi_j)_{j+1;m,\bar{m}} = N_j\bar{N}_j \sum_{r,s=-1}^{1} c_r(j,m)c_s(j,\bar{m})\psi^r\bar{\psi}^s\Phi_{j;m-r,\bar{m}-s}\,, \tag{2.6}$$

where we use the notation $(\psi^1,\psi^0,\psi^{-1}) \equiv (\psi^+,\psi^3,\psi^-)$ and the Clebsch-Gordan coefficients $c_r(j,m)$ are given by

$$\begin{aligned} c_1(j,m) &= -\frac{1}{\sqrt{2}}(j+m)(j+m+1)\,, \\ c_0(j,m) &= (j+m+1)(j-m+1)\,, \\ c_{-1}(j,m) &= \frac{1}{\sqrt{2}}(j-m)(j-m+1)\;. \end{aligned} \tag{2.7}$$

The Clebsch-Gordan coefficients are determined in terms of the coefficients in the action of $J^{\pm}$ on the primaries $\Phi_{j;m}$. In our case they differ from the more familiar form involving square roots due to our conventions in (2.5). We have also introduced a convenient j-dependent normalization factor given by

$$N_j = \bar{N}_j = \frac{1}{(2j+1)(2j+2)}\;. \tag{2.8}$$

2.2 Chiral and antichiral primaries

At this stage one could ask if the CFT operators in (2.1) have any special properties. Since the CHS background exhibits $\mathcal{N}=4$ superconformal invariance, a natural question is if there are any chiral or antichiral primaries among them. Let us choose the $\mathcal{N}=2$ subalgebra generated by the energy-momentum tensor

$$T = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}q\partial^2\phi + \frac{J^iJ^i}{k} - \frac{1}{2}\psi^*\partial\psi - \frac{1}{2}\psi\partial\psi^* - \frac{1}{2}\psi^+\partial\psi^- - \frac{1}{2}\psi^-\partial\psi^+\,, \tag{2.9}$$

the supercurrents

$$\begin{aligned} G^+ &= i\psi\left(\partial\phi - qJ_3 - q\psi^+\psi^-\right) + iq\partial\psi + qJ^-\psi^+\,, \\ G^- &= i\psi^*\left(\partial\phi + qJ_3 + q\psi^+\psi^-\right) + iq\partial\psi^* + qJ^+\psi^-\,, \end{aligned} \tag{2.10}$$

and the U(1) R-current

$$J_R = \psi\psi^* + \psi^+\psi^- = -i\psi_\phi\psi_3 + \psi^+\psi^-\;. \tag{2.11}$$

The fermion combinations

$$\psi^{\pm} = \frac{1}{\sqrt{2}}(\psi_1 \pm i\psi_2)\,, \qquad \psi = \frac{1}{\sqrt{2}}(\psi_\phi + i\psi_3)\,, \tag{2.12}$$

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with ψ_ϕ being the superpartner of ϕ, satisfy the following operator product expansions

$$\psi(z)\psi^*(w) = \psi^+(z)\psi^-(w) \sim \frac{1}{z-w} \ . \tag{2.13}$$

For considerations of spacetime supersymmetry, it will be also useful to bosonize the above fermions as

$$\psi^\pm = e^{\pm iH_1} , \quad \psi = e^{iH_2} \tag{2.14}$$

with H_1 and H_2 being canonically normalized bosons with OPEs

$$H_1(z)H_1(w) = H_2(z)H_2(w) \sim -\ln(z-w) , \quad H_1(z)H_2(w) = 0 \ . \tag{2.15}$$

Then the R-current can be written as

$$J_R = i\partial H_1 + i\partial H_2 \tag{2.16}$$

and (half of) the spacetime supercharges, which live on the 5+1-dimensional worldvolume of the NS5-branes, are given by

$$Q_\alpha^\pm = \frac{1}{2\pi i}\oint dz e^{-\frac{\varphi}{2}\pm\frac{i}{2}(H_1+H_2)}\mathcal{S}_\alpha \ . \tag{2.17}$$

In this formula φ stands for the bosonized superconformal ghosts and $\mathcal{S}_\alpha$ are worldvolume spin fields in the $\mathbf{4}$ of $\mathrm{SO}(5,1)$ whose explicit form will not be necessary. For NS5-branes in type II theories a similar set of spacetime supercharges arises from the antiholomorphic sector, so that all together we have 16 spacetime supersymmetries. Notice that in general we will focus only on the holomorphic sector, since exactly the same expressions hold for the antiholomorphic one, and from now on we will suppress in most formulas all antiholomorphic indices to avoid cluttering.

Recall that a field χ is primary of the $\mathcal{N}=2$ superconformal algebra if it satisfies

$$\begin{aligned} T(z)\chi(w) &\sim \frac{h}{(z-w)^2}\chi(w) + \frac{\partial\chi(w)}{z-w} , \\ J_R(w)\chi(w) &\sim \frac{q}{z-w}\chi(w) , \\ G^\pm(z)\chi(w) &\sim \frac{1}{z-w}\widetilde{\chi}^\pm(w) , \end{aligned} \tag{2.18}$$

with h being its conformal weight and q its U(1) R-charge. In addition, it is chiral (antichiral) if its OPE with the supercurrent $G^+(z)$ $(G^-(z))$ is regular [8]. When a field is both chiral (antichiral) and primary its conformal dimension is fixed in terms of its U(1) R-charge as $h = |q|/2$. As a consequence h is not renormalized as long as superconformal invariance remains unbroken. In particular, chiral and antichiral primary operators with $|q| = 1$ which yield marginal deformations when acted with the $\mathcal{N}=1$ supercharge $G = \frac{1}{\sqrt{2}}(G^+ + G^-)$, actually give rise to *exactly marginal* deformations.

It is a straightforward exercise to check that $(\psi\Phi_j)_{j+1;m}e^{-qa_j\phi}$ is a superconformal primary when $m = j+1$ or $m = -j-1$. Then, the corresponding operators take the form $\psi^+\Phi_{j;j}e^{-q\alpha_j\phi}$ and $\psi^-\Phi_{j;-j}e^{-q\alpha_j\phi}$, respectively. Notice that these superconformal primaries are built on affine primaries $\Phi_{j,\pm j}$ of the $\mathrm{SU}(2)_{k-2}$ WZW model. Furthermore, out of the class of operators $(\psi\Phi_j)_{j+1;m}e^{-qa_j\phi}$, only $\psi^+\Phi_{j;m-1}e^{-qa_j\phi}$ and $\psi^-\Phi_{j;m+1}e^{-qa_j\phi}$ can have special chirality properties. These operators are chiral (antichiral) when $m = a_j$ ($m = -a_j$). The final conclusion is that we have a set of chiral primaries given by $\psi^+\Phi_{j;j}e^{-q(j+1)\phi}$ along with their conjugates $\psi^-\Phi_{j;-j}e^{-q(j+1)\phi}$ which are antichiral primaries. It is interesting to note that only normalizable operators, i.e. with $a_j = j+1$, can be chiral or antichiral primaries.

Another operator we could consider is $\psi\Phi_{j;m}e^{-qa_j\phi}$ and its conjugate, although only their real part $\psi_3\Phi_{j;m}e^{-qa_j\phi}$ appears in the holographic dictionary. These operators are primary when $m = 1 - a_j$ and they are chiral (antichiral) when $m = -j$ ($m = j$). Hence we conclude that $\psi\Phi_{j;-j}e^{-q(j+1)\phi}$ $\big(\psi^*\Phi_{j;j}e^{-q(j+1)}\big)$ is a chiral (antichiral) primary. Note that their non-normalizable counterparts are not chiral or antichiral primaries.

The fact that the non-normalizable versions of the chiral (antichiral) primaries are not also chiral (antichiral) primaries seems a bit puzzling at first sight. For instance, although $\psi^+\Phi_{j;j}e^{-q(j+1)\phi}$ and $\psi\Phi_{j;-j}e^{-q(j+1)\phi}$ are chiral primary (and their conjugates antichiral primary), their non-normalizable versions, that share the same h and q, are not. This seems to violate the standard argument that an operator with $h = q/2$ ($h = -q/2$) is chiral (antichiral) primary. This argument is based on the observation that [8]

$$\langle\chi|\{G^+_{-\frac{1}{2}}, G^-_{\frac{1}{2}}\}|\chi\rangle = \langle\chi|2L_0 - (J_R)_0|\chi\rangle = (2h-q)\langle\chi|\chi\rangle\,, \tag{2.19}$$

where the $\mathcal{N}=2$ superconformal algebra was used. If $h = q/2$ one gets $\langle\chi|\{G^+_{-\frac{1}{2}}, G^-_{\frac{1}{2}}\}|\chi\rangle = 0$ and using hermiticity of the supercurrents $(G^{\pm}_r)^\dagger = G^{\mp}_{-r}$ along with positivity of the inner product leads to $G^+_{-\frac{1}{2}}|\chi\rangle = G^-_{\frac{1}{2}}|\chi\rangle = 0$. The resolution of the puzzle is that the linear dilaton CFT contains non-unitary representations that correspond to fields with negative conformal weights.

For instance, $\psi^+\Phi_{j;j}e^{qj\phi}$ is non-chiral and therefore, if $|\chi\rangle$ is the corresponding state, we have $G^+_{-\frac{1}{2}}|\chi\rangle \neq 0$. Indeed, we obtain

$$|\hat{\chi}\rangle = G^+_{-\frac{1}{2}}|\chi\rangle = -iq(2j+1)\psi\psi^+\Phi_{j;j}e^{qj\phi}|\Omega\rangle\,, \tag{2.20}$$

where $|\Omega\rangle$ is the vacuum. This state, however, satisfies $G^-_{\frac{1}{2}}|\hat{\chi}\rangle = 0$ since there is no second order pole between $G^-(z)$ and the operator $\psi\psi^+\Phi_{j;j}e^{qj\phi}(w)$. Hence, although the operator $\psi^+\Phi_{j;j}e^{qj\phi}$ is non-chiral, it still has $h = q/2 = 1/2$ and (2.19) is obeyed. Similarly, $\psi\Phi_{j;-j}e^{qj\phi}$ is chiral but not primary since the state $|\zeta\rangle$ it creates is not annihilated by $G^-_{\frac{1}{2}}$:

$$|\hat{\zeta}\rangle = G^-_{\frac{1}{2}}|\zeta\rangle = -iq(2j+1)\Phi_{j;-j}e^{qj\phi}|\Omega\rangle\ . \tag{2.21}$$

However, we can again check that $G^+_{-\frac{1}{2}}|\hat{\zeta}\rangle = 0$ since the OPE of $G^+(z)$ with $\Phi_{j;-j}e^{qj\phi}$ is regular. The existence of non-unitary representations of the linear dilaton CFT underlies both effects as it is obvious from the fact that the zero-norm states $|\hat{\chi}\rangle$ and $|\hat{\zeta}\rangle$ vanish when the background charge q is zero.

2.3 Supersymmetric deformations

According to the prescription given in [5, 6], when the operators $\widetilde{\mathrm{tr}}(X^{i_1}X^{i_2}\cdots X^{i_{2j+2}})$ obtain non-zero VEVs the original Lagrangian $\mathcal{L}_0$ of the holographically dual conformal field theory is perturbed to

$$\mathcal{L} = \mathcal{L}_0 + \sum_{j=0}^{\frac{k-2}{2}} \sum_{m,\bar{m}=-(j+1)}^{j+1} \left(\lambda_{j;m,\bar{m}} G_{-\frac{1}{2}} \bar{G}_{-\frac{1}{2}} (\psi\bar{\psi}\Phi_j^{\mathrm{su}})_{j+1;m,\bar{m}} e^{-q(j+1)\phi} + \text{c.c.}\right) . \tag{2.22}$$

Here $G(z)$ is the $\mathcal{N}=1$ supercurrent

$$G = i\psi_\phi \partial\phi + q\psi_3 J_3 + q\psi_3\psi^+\psi^- + iq\partial\psi_\phi + \frac{q}{\sqrt{2}}(J^-\psi^+ + J^+\psi^-) , \tag{2.23}$$

and the couplings $\lambda_{j;m,\bar{m}}$ are specified in terms of $\widetilde{\mathrm{tr}}(X^{i_1}X^{i_2}\cdots X^{i_{2j+2}})$ in a way that we will make precise in the next section. Notice that by construction the deformation preserves $\mathcal{N}=(1,1)$ superconformal invariance.

The action of $G_{-\frac{1}{2}}$ can be read from the simple pole of $G(z)$ in its OPE with $(\psi\Phi_j)_{j+1;m} e^{-qa_j\phi}$ and it yields a piece without fermions and a piece bilinear in the fermions. The first piece reads

$$qN_j \sum_{r=-1}^{1} c_r(j,m)\lambda_r J^r \Phi_{j;m-r} e^{-a_j q\phi} , \tag{2.24}$$

where $(J^{\pm 1}, J^0) \equiv (J^\pm, J^3)$ and $\lambda_0 = 1, \lambda_{\pm 1} = \frac{1}{\sqrt{2}}$. The fermion bilinear term is

$$qN_j\left(\left(\sum_{r=-1}^{1} ia_j c_r \psi_\phi \psi^r \Phi_{j;m-r}\right) + d_1\psi_3\psi^+\Phi_{j;m-1} + d_{-1}\psi_3\psi^-\Phi_{j;m+1} + d_0\psi^+\psi^-\Phi_{j;m}\right) e^{-a_j q\phi} , \tag{2.25}$$

where we defined the combinations

$$\begin{aligned} d_{\pm 1} &= mc_{\pm 1} - \frac{c_0}{\sqrt{2}}(j \pm m) , \\ d_0 &= c_0 + \frac{1}{\sqrt{2}}\Big(c_{-1}(j+m+1) - c_1(j-m+1)\Big) . \end{aligned} \tag{2.26}$$

Notice that one could reverse the logic and start with an ansatz for the deformation that is the sum of (2.24) and (2.25) with arbitrary coefficients c_r and d_r. Then, the equations (2.26) could be thought of as conditions for preserving $\mathcal{N}=1$ worldsheet supersymmetry.

2.3.1 $\mathcal{N}=2$ supersymmetry

We will first uncover the conditions for the sum of the deformations (2.24) and (2.25) to preserve $\mathcal{N}=2$ supersymmetry and then extend the analysis to $\mathcal{N}=4$. If $\mathcal{N}=2$ is preserved, the deformation should be annihilated by both supercharges $G^\pm_{-\frac{1}{2}}$ or, equivalently, by $G_{-\frac{1}{2}}$ and $G^3_{-\frac{1}{2}}$ where

$$G^3 = -\frac{i}{\sqrt{2}}(G^+ - G^-) . \tag{2.27}$$

Since the deformation we consider arises from the action of $G_{-\frac{1}{2}}$ on $(\psi\Phi_j)_{j+1;m}e^{-qa_j\phi}$, it is automatically annihilated by $G_{-\frac{1}{2}}$. Furthermore, a sufficient condition for the deformation being annihilated by $G^3_{-\frac{1}{2}}$ is that it has zero R-charge, as can be seen by the following $\mathcal{N}=2$ commutation relation

$$[(J_R)_0, G_{-\frac{1}{2}}] = iG^3_{-\frac{1}{2}} \ . \tag{2.28}$$

Actually, if we were only interested in preserving $\mathcal{N}=2$ supersymmetry, it would be enough to just demand definite R-charge. However, we want to preserve $\mathcal{N}=2$ superconformal invariance and since the R-symmetry is part of the $\mathcal{N}=2$ SCFT algebra, the deformations we consider have to be neutral. Notice also that the condition we just formulated is not necessary and, in principle, it could miss some supersymmetric deformations. However, we will soon establish that for the operators under consideration it is actually necessary, besides being sufficient.

The purely bosonic part (2.24) of the deformation obviously carries zero charge under the R-current (2.11). Instead, the fermionic piece (2.25) has zero charge only when the following conditions are satisfied

$$d_{\pm 1} = \pm a_j c_{\pm 1} \ . \tag{2.29}$$

For a normalizable operator, which means we select $a_j = j+1$, these conditions are satisfied automatically. As we will see soon, these operators preserve also supersymmetry.

Of course it is expected that chiral or antichiral operators yield deformations that preserve $\mathcal{N}=2$ supersymmetry. Indeed, a chiral primary state $|\chi\rangle$ satisfies $G^+_{-\frac{1}{2}}|\chi\rangle = 0$ and hence the deformation it yields is $|\widetilde{\chi}\rangle = G_{-\frac{1}{2}}|\chi\rangle = \frac{1}{\sqrt{2}}G^-_{-\frac{1}{2}}|\chi\rangle$. This state is obviously annihilated by $G^-_{-\frac{1}{2}}$ and furthermore, using the $\mathcal{N}=2$ commutation relation $\{G^+_{-\frac{1}{2}}, G^-_{-\frac{1}{2}}\} = 2L_{-1}$, we see that it is also annihilated by $G^+_{-\frac{1}{2}}$ up to a total derivative that does not affect the action. Similarly, the sum of a chiral and an antichiral operator yields again a $\mathcal{N}=2$ supersymmetric deformation. For instance the operator with $j=m=0$ belongs to this category. However, not all deformations preserving $\mathcal{N}=2$ supersymmetry need originate from a chiral or antichiral operator. For instance, all operators with $|m| \neq (j+1)$ yield $\mathcal{N}=2$ preserving deformations but none of them is chiral or antichiral primary.

For non-normalizable operators with $a_j = -j$ there is only one solution of the $\mathcal{N}=2$ constraints (2.29) given by $j=m=0$. The corresponding operator is ψ^3 and it leads to the deformation $J^3 + \psi^+\psi^-$. We hasten to point out that although this deformation preserves extended worldsheet supersymmetry, it does not lead to a spacetime supersymmetric background[2] since it does not commute with the spacetime supercharges (2.17).

Let us now check that the above argument, based on R-charge neutrality, does not miss any solutions. This can be done by examining explicitly some terms of the OPE of $G^3(z)$ with the deforming operator. Explicitly, this supercurrent is

$$G^3 = i\psi_3\partial\phi - q\psi_\phi J_3 - q\psi_\phi\psi^+\psi^- + iq\partial\psi_3 + i\frac{q}{\sqrt{2}}(J^+\psi^- - J^-\psi^+) \tag{2.30}$$

[2] Interesting applications of that mechanism of supersymmetry breaking in non-critical superstrings can be found in [9–11].

and let us keep only the terms of its OPE with the sum of (2.24) and (2.25) containing ψ_3. These terms read

$$\frac{iq}{z-w}\Bigg(a_j \sum_{r=-1}^{1} c_r \lambda_r \psi_3 J^r \Phi_{j;m-r} e^{-a_j q\phi} - a_j c_0 \psi_3 J^3 \Phi_{j;m} \\ - \frac{d_{+1}}{\sqrt{2}} \psi_3 J^+ \Phi_{j;m-1} + \frac{d_{-1}}{\sqrt{2}} \psi_3 J^- \Phi_{j;m+1} \Bigg) \tag{2.31}$$

and they vanish if and only if (2.29) are satisfied. Hence, for the class of operators under consideration, the condition of vanishing R-charge is not only sufficient but also necessary for preserving $\mathcal{N}=2$ supersymmetry.

2.3.2 $\mathcal{N}=4$ supersymmetry

Similarly to the $\mathcal{N}=2$ case, a sufficient condition for preserving $\mathcal{N}=4$ SCFT invariance is that the deformation is a singlet under the corresponding R-symmetry group $SU(2)_R$. The latter is generated by J_R and two more generators $S^{\pm}$:

$$\mathrm{SU}(2)_R: \qquad J_R = \psi^+\psi^- + \psi\psi^*, \quad S^+ = \psi\psi^+, \quad S^- = \psi^-\psi^* \ . \tag{2.32}$$

The OPEs of $S^{\pm}(z)$ with (2.25) are zero provided that besides (2.29), which means that we already assume preservation of $\mathcal{N}=2$, the following condition is satisfied

$$d_0 = a_j c_0 \ . \tag{2.33}$$

This condition holds automatically for all operators in the normalizable branch that preserve $\mathcal{N}=2$. Let us present their fermionic pieces for completeness:

$$qN_j(j+1)\Big(i\sqrt{2}(c_1\psi^*\psi^+\Phi_{j;m-1} + c_{-1}\psi\psi^-\Phi_{j;m+1}) + c_0(-\psi\psi^* + \psi^+\psi^-)\Phi_{j;m}\Big)e^{-q(j+1)\phi} \ . \tag{2.34}$$

It is easy also to establish that these deformations preserve spacetime supersymmetry by writing the above fermion bilinears in bosonized form

$$\psi^*\psi^+ = e^{-iH_2+iH_1}, \quad \psi\psi^- = e^{iH_2-iH_1}, \quad -\psi\psi^* + \psi^+\psi^- = -i\partial H_2 + i\partial H_1, \tag{2.35}$$

where it is manifest that they commute with the spacetime supercharges (2.17). Actually since the same combination of H_1 and H_2 appears in $S^{\pm}$ and in the spacetime supercharges, we conclude that any deformation preserving $\mathcal{N}=4$ superconformal invariance automatically preserves spacetime supersymmetry as well. As an example, we notice that the usual marginal deformation $J^3\bar{J}^3$ of the bosonic SU(2) WZW model can be promoted to an operator in the CHS background that preserves $\mathcal{N}=(4,4)$ superconformal invariance in the following way

$$(J^3 - \psi\psi^* + \psi^+\psi^-)(\bar{J}^3 - \bar{\psi}\bar{\psi}^* + \bar{\psi}^+\bar{\psi}^-)e^{-q\phi} \ . \tag{2.36}$$

Going now over to the non-normalizable sector, we observe that the unique such operator preserving $\mathcal{N}=2$, the one with $j=m=0$, does not satisfy (2.33) and hence does not preserve $\mathcal{N}=4$. Hence, the non-normalizable deformation $(J^3+\psi^+\psi^-)(\bar{J}^3+\bar{\psi}^+\bar{\psi}^-)$ preserves only $\mathcal{N}=(2,2)$ supersymmetry (but not any spacetime supersymmetry as we emphasized earlier).

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2.4 More supersymmetric deformations

In this subsection we investigate the possibility that other classes of operators lead to supersymmetric marginal deformations. One question is under what conditions a deformation that originates from the operator

$$\mu_3\psi^3\Phi_{j;m_3}e^{-a_jq\phi}+\mu_+\psi^+\Phi_{j;m_+}e^{-a_jq\phi}+\mu_-\psi^-\Phi_{j;m_-}e^{-a_jq\phi}\,, \tag{2.37}$$

preserves $\mathcal{N}=2$ and $\mathcal{N}=4$ superconformal invariance. This operator differs from $(\psi\Phi_j)_{j+1;m}e^{-qa_j\phi}$ since the coefficients $\mu_\pm,\mu_3$ are arbitrary and we do not assume a priori any relation between m_3 and $m_\pm$. As before, the deformation arises by the action of $G_{-\frac{1}{2}}\bar{G}_{-\frac{1}{2}}$ on (2.37) and hence $\mathcal{N}=(1,1)$ supersymmetry is guaranteed by construction. Notice that we consider a single j since there cannot be any mixing among different j's upon the action of the supercharges.

The deformations that arise from each of the three operators in (2.37) are:

$$q\mu_3\Bigg(J_3\Phi_{j;m_3}+ia_j\psi_\phi\psi^3\Phi_{j;m_3}+\psi^+\psi^-\Phi_{j;m_3}+\frac{1}{\sqrt{2}}(j+m_3)\psi^+\psi^3\Phi_{j;m_3-1}$$
$$+\frac{1}{\sqrt{2}}(j-m_3)\psi^-\psi^3\Phi_{j;m_3+1}\Bigg)e^{-a_jq\phi}\,, \tag{2.38}$$

$$q\mu_+\Bigg(\frac{1}{\sqrt{2}}J^+\Phi_{j;m_+}+ia_j\psi_\phi\psi^+\Phi_{j;m_+}+(m_++1)\psi^3\psi^+\Phi_{j;m_+} \tag{2.39}$$
$$+\frac{1}{\sqrt{2}}(j-m_+)\psi^-\psi^+\Psi_{j;m_++1}\Bigg)e^{-a_jq\phi}\,,$$

and

$$q\mu_-\Bigg(\frac{1}{\sqrt{2}}J^-\Phi_{j;m_-}+ia_j\psi_\phi\psi^-\Phi_{j;m_-}+(m_--1)\psi^3\psi^-\Phi_{j;m_-} \tag{2.40}$$
$$+\frac{1}{\sqrt{2}}(j+m_-)\psi^+\psi^-\Phi_{j;m_--1}\Bigg)e^{-a_jq\phi}\,.$$

As explained previously, a sufficient condition for preserving $\mathcal{N}=2$ supersymmetry is that these deformations are neutral under the U(1) R-current J_R. For the example under study we find that neutrality under J_R is guaranteed if the following conditions are satisfied:

$$\begin{aligned}\frac{1}{\sqrt{2}}\mu_3(j+m_3)\Phi_{j;m_3-1}+\mu_+(a_j-m_+-1)\Phi_{j;m_+}&=0\,,\\ \frac{1}{\sqrt{2}}\mu_3(j-m_3)\Phi_{j;m_3+1}-\mu_-(a_j+m_--1)\Phi_{j;m_-}&=0\,.\end{aligned} \tag{2.41}$$

Furthermore, in order to ensure $\mathcal{N}=4$ invariance we need to check that the deformations are also neutral under the extra generators $S^\pm$ which, along with J_R, generate the R-symmetry group $\mathrm{SU}(2)_R$ of the $\mathcal{N}=4$ superconformal algebra. We find three conditions. Two of those are identical with these that guarantee $\mathcal{N}=2$ invariance. This is expected since the $S^\pm$ generators close on J_R. The third condition reads

$$\mu_3(a_j-1)\Phi_{j;m_3}+\frac{1}{\sqrt{2}}\mu_+(j-m_+)\Phi_{j;m_++1}-\frac{1}{\sqrt{2}}\mu_-(j+m_-)\Phi_{j;m_--1}=0\,. \tag{2.42}$$

Equations (2.41) and (2.42) provide a set of sufficient conditions for the deformation (2.38)+(2.40)+(2.40) to preserve $\mathcal{N}=4$ supersymmetry. These conditions yield different equations for the coefficients $\mu_3, \mu_\pm$ depending on whether the charges m_3 and $m_\pm$ are related or not. The simplest case to analyze is that of $m_3 = m_+ + 1 = m_- - 1 = m$. Then, the $\mathcal{N}=2$ conditions fix $\mu_\pm$ in terms of μ_3 as

$$\mu_\pm = \mp \frac{1}{\sqrt{2}} \frac{j \pm m}{a_j \mp m}\, \mu\,, \qquad \mu_3 = \mu\ . \tag{2.43}$$

The $\mathcal{N}=4$ condition yields a further constraint

$$\mu_3(a_j - 1) + \frac{1}{\sqrt{2}}\mu_+(j - m + 1) - \frac{1}{\sqrt{2}}\mu_-(j + m + 1) = 0\ . \tag{2.44}$$

Upon combining with (2.43) we find that in order to have non-trivial solutions a_j has to equal $a_j = -j, j+1, 0$. In other words, the values of a_j that are singled-out by $\mathcal{N}=4$ supersymmetry include those for which the deformation is marginal. Instead, the case of $a_j = 0$ (with the exception of $j = 0$ which is analyzed below) leads to an irrelevant operator.

For normalizable deformations, i.e. $a_j = j+1$, the solution (2.43) yields the class of holographic operators $(\psi\Phi_j)_{j+1;m} e^{-q(j+1)\phi}$. As we already know these operators preserve $\mathcal{N}=4$ supersymmetry and hence it is not necessary to check (2.42) (it is automatically satisfied). The purely bosonic part of the corresponding deformation is

$$q\left(\mu_3 J^3 \Phi_{j;m} + \frac{\mu_+}{\sqrt{2}} J^+ \Phi_{j;m-1} + \frac{\mu_-}{\sqrt{2}} J^- \Phi_{j,m+1}\right) e^{-q(j+1)\phi}\,, \tag{2.45}$$

with $|m| \leqslant j+1$ and with the coefficients being given by

$$\mu_\pm = \mp \frac{1}{\sqrt{2}} \frac{j \pm m}{j + 1 \mp m}\, \mu\,, \qquad \mu_3 = \mu\ . \tag{2.46}$$

Choosing $\mu = (j+1-m)(j+m+1)$ we obtain indeed the Clebsch-Gordan coefficients (2.7).

For non-normalizable operators, i.e. $a_j = -j$, the solution (2.43) boils down to

$$\mu_\pm = \pm \frac{1}{\sqrt{2}}\mu\,, \qquad \mu_3 = \mu\,, \tag{2.47}$$

which satisfies also the $\mathcal{N}=4$ condition (2.44). This non-normalizable solution is by far more general than the one found in the previous subsection, where μ_3 and $\mu_\pm$ were specified in terms of the Clebsch-Gordan coefficients fixing $j = m = 0$. Instead, the current solution exists for any values of j and m that are allowed. We will denote from now on the corresponding operator in (2.37) by $(\psi\Phi_j)_m e^{qj\phi}$ since it does not have definite spin (it does have definite $\mathcal{J}^3$ charge though). The associated bosonic deformation reads

$$q\mu\left(J^3 \Phi_{j;m} + \frac{1}{2} J^+ \Phi_{j;m-1} - \frac{1}{2} J^- \Phi_{j;m+1}\right) e^{qj\phi} \tag{2.48}$$

and the fermion bilinear piece is

$$qj\mu\Big(i\psi\psi^-\Phi_{j;m+1}+i\psi^*\psi^+\Phi_{j;m-1}+(\psi\psi^*-\psi^+\psi^-)\Phi_{j;m}\Big)e^{qj\phi}\ . \tag{2.49}$$

Since these deformations preserve $\mathcal{N}=4$ superconformal invariance, they also preserve spacetime supersymmetry. Notice that these operators, for generic j and m, do not have a holographic counterpart since they do not have definite total spin, and therefore cannot correspond to an LST deformation. Their interpretation in terms of NS5-branes will be uncovered in section 4.

There are two more classes of operators that lead automatically to $\mathcal{N}=2$ preserving deformations. These are $\psi\Phi_{j;-j}e^{-q(j+1)}\phi$ and $\psi^*\Phi_{j;j}e^{-q(j+1)}$ which, as was shown in subsection 2.2, are chiral and antichiral primaries, respectively. They were not captured by the analysis we just performed since the ansatz (2.37) does not contain the fermion ψ_ϕ. It can be checked that the corresponding deformations preserve also $\mathcal{N}=4$ supersymmetry and therefore spacetime supersymmetry. The purely bosonic piece of the deformation coming from $\psi\Phi_{j;-j}e^{-q(j+1)}\phi$ is

$$\frac{i}{\sqrt{2}}(\partial\phi+qJ_3)\Phi_{j;-j}e^{-q(j+1)\phi} \tag{2.50}$$

and the fermion bilinear piece is

$$\frac{q}{\sqrt{2}}\left(i\psi^+\psi^-\Phi_{j;-j}-i\psi\psi^*\Phi_{j;-j}+2j\psi^-\psi\Phi_{j;-j+1}\right)e^{-q(j+1)\phi}\ . \tag{2.51}$$

Notice that $\psi\Phi_{j;-j}e^{-q(j+1)}\phi$ and $\psi^*\Phi_{j;j}e^{-q(j+1)}$ do not have definite spin under the spacetime symmetry SO(4) and do not appear independently in the holographic dictionary (only their imaginary part for $j=0$, which is $\psi_3e^{-q\phi}$, does have a holographic interpretation). In that respect, they are similar to the non-normalizable operators we discussed earlier, which also preserve worldsheet and spacetime supersymmetry. Although both classes of operators leave intact the 6-dim Lorentz invariance associated with the worldvolume of the NS5-branes, they lack an interpretation in LST.

The non-normalizable versions of $\psi\Phi_{j;-j}e^{-q(j+1)}\phi$ and $\psi^*\Phi_{j;j}e^{-q(j+1)}$ are chiral and antichiral, respectively, but not primary. They are examples of operators where $h=|q|/2$ but due to non-unitarity they fail to be chiral primary. Furthermore, it can be checked that they preserve $\mathcal{N}=2$ supersymmetry but not $\mathcal{N}=4$ and hence they break spacetime supersymmetry.

An interesting observation is that the extra operators $\psi\Phi_{j;-j}e^{-q(j+1)}\phi$ and $\psi^*\Phi_{j;j}e^{-q(j+1)}$ are actually BRST trivial in the $\mathcal{N}=2$ topologically twisted theory. The reason is that they arise from the action of $G^+(z)$ on $\Phi_{j;-j}e^{-q(j+1)\phi}$ since

$$G^+(z)\Phi_{j;-j}e^{-q(j+1)\phi}(w)\sim\frac{iq(2j+1)}{z-w}\psi\Phi_{j;-j}e^{-q(j+1)\phi}(w) \tag{2.52}$$

(and similarly for the complex conjugate). In the topological theory where the energy momentum tensor is $T(z)+\frac{1}{2}\partial J_R(z)$ the BRST charge is $Q_{\rm BRST}=\oint G^+(z)dz$ and $\psi\Phi_{j;-j}e^{-q(j+1)\phi}$ is a trivial element of the BRST cohomology.

Let us also point out that the holographic operators $(\psi\Phi_j)_{j+1;m}e^{-q(j+1)\phi}$ originate from the action of the extra two $\mathcal{N}=4$ supercharges on $\Phi_{j;-j}e^{-q(j+1)\phi}$. These supercurrents read

$$\tilde{G}^+ = i\psi^+\big(\partial\phi + qJ_3 - q\psi\psi^*\big) + iq\partial\psi^+ - qJ^+\psi\,, \tag{2.53}$$
$$\tilde{G}^- = i\psi^-\big(\partial\phi - qJ_3 + q\psi\psi^*\big) + iq\partial\psi^- - qJ^-\psi^*\,, \tag{2.54}$$

and, along with $G^{\pm}(z)$, generate the $\mathcal{N}=4$ superconformal algebra. Then it holds that

$$\tilde{G}^+(z)\Phi_{j;j}e^{-q(j+1)\phi}(w) \sim \frac{iq(2j+1)}{z-w}\psi^+\Phi_{j;j}e^{-q(j+1)\phi}(w)\ . \tag{2.55}$$

Hence, with respect to the $\mathcal{N}=2$ algebra generated by $\tilde{G}^{\pm}(z)$, the operator $\psi^+\Phi_{j;j}e^{-q(j+1)\phi}$ would be BRST trivial after the topological twisting. However, neither $\psi\Phi_{j;-j}e^{-q(j+1)\phi}$ nor $\psi^+\Phi_{j;j}e^{-q(j+1)\phi}$ are trivial as elements of the BRST cohomology of the $\mathcal{N}=4$ topological string.

When $j=m=0$ the real part of $\psi\Phi_{j;-j}e^{-q(j+1)\phi}$ and of its non-normalizable version preserves $\mathcal{N}=4$ but the deformation it leads to, whose purely bosonic piece reads $\partial\phi\bar{\partial}\phi e^{-qa_0\phi}$, is trivial since it is tantamount to a coordinate redefinition of the linear dilaton direction. This triviality, however, does not seem to persist when $j\neq 0$ since primaries of the $\mathrm{SU}(2)_{k-2}$ WZW model couple to the derivatives of the dilaton.

2.5 Comments and summary

Normalizable CFT operators of the form $(\psi\bar{\psi}\Phi_j)_{j+1;m,\bar{m}}e^{-q(j+1)\phi}$ correspond holographically to VEVs of the operators $\widetilde{\mathrm{tr}}(X^{i_1}X^{i_2}\cdots X^{i_{2j+2}})$ that parametrize the moduli space of LST. Notice that spacetime supersymmetry does not change as we move in the moduli space since any configuration of parallel NS5-branes, irrespectively of their positions in the transverse space, preserves 16 supercharges in type II theories. Consequently, the $\mathcal{N}=4$ superconformal symmetry of the original underlying CFT should also be left intact [12]. We have shown that all deformations originating from $(\psi\bar{\psi}\Phi_j)_{j+1;m,\bar{m}}e^{-q(j+1)\phi}$ preserve both $\mathcal{N}=(4,4)$ worldsheet supersymmetry and 16 spacetime supercharges. Therefore, all of those that bear non-vanishing couplings $\lambda_{j;m,\bar{m}}$ can be in principle present in the deformed Lagrangian.

The last observation is particularly puzzling for two reasons. First, as noticed in [5], there is a mismatch between the number of couplings $\lambda_{j;m,\bar{m}}$, which grows as k^3, and the number of parameters that determine a point in the moduli space of LST, the latter being $4(k-1)$. Second, [7] established that the most general planar deformation of the NS5-branes[3] was captured by a subset of the possible deforming operators, namely those that are (chiral, chiral) primaries as well as their (antichiral, antichiral) conjugates. It is also quite straightforward to see that non-planar deformations of the NS5-brane are captured by (chiral, antichiral) and (antichiral, chiral) operators.

[3]Notice that [7] considered deformations of a circular distribution of NS5-branes where the underlying CFT is $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)\times\mathrm{SU}(2)_k/\mathrm{U}(1)$. As we mentioned already, there is a 1-1 correspondence between operators in that theory and the CHS theory studied here, so that all results pertaining to $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)\times\mathrm{SU}(2)_k/\mathrm{U}(1)$ deformations can be rephrased in the CHS theory.

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Therefore, it seems that the chiral ring operators, namely those with $m, \bar{m} = \pm(j+1)$, are sufficient to capture the most general geometric NS5-brane deformation. Then, the puzzle raised in [5] is resolved since the number of such operators is precisely $4(k-1)$ (recall that $j = 0, \frac{1}{2}, \ldots, \frac{k-2}{2}$ and we have to combine the holomorphic with the antiholomorphic part). This is also in line with the fact that in the T-dual theory, which is described by a σ model with an ALE target space and without the complications due to the presence of NS-NS flux, only operators in the chiral ring correspond to geometric moduli. Therefore, we will also dub "non-holographic" all operators of the form $(\psi\bar{\psi}\Phi_j)_{j+1;m,\bar{m}}e^{-q(j+1)\phi}$ with $|m|$ or $|\bar{m}|$ different than $j+1$.

One extra argument in support of this proposal is that the marginal deformations originating from chiral (or antichiral) operators are actually exactly marginal. As we said earlier, the reason is that these operators have protected conformal dimensions since the latter are fixed in terms of the non-renormalized $\mathrm{U}(1)$ R-charge as $h = \frac{q}{2}$. Since the NS5-branes can be finitely separated without spoiling the conformal invariance of the worldsheet theory, we are lead to the conclusion that only exactly marginal deformations, which in principle can be integrated to finite deformations, should be used to perturb the original CFT.

Notice that deformations originating from non-chiral operators are not, in general, exactly marginal since they do not satisfy the criterion of [13]. However, there is an exception provided by the operator $j = m = \bar{m} = 0$. The purely posonic piece of the corresponding deformation is $J^3\bar{J}^3e^{-q\phi}$ and it is well-known [13] that $J^3\bar{J}^3$ is an exactly marginal operator of the $\mathrm{SU}(2)_{k-2}$ WZW model. The Liouville dressing does not modify the argument of [13] since the operator $e^{-q\phi}$ is equivalent to the identity operator in Liouville theory and its OPE with itself is trivial.

Therefore, we propose that the normalizable operator with $j = m = \bar{m} = 0$ should also be taken into account when one considers NS5-brane deformations. An additional reason for doing so is that in the simple example where the point-like configuration of NS5-brane is deformed to a circle, this operator yields the leading deformation of the CHS theory (or, more precisely, of its T-dual) towards the model $\mathrm{SL}(2,\mathbb{R})_k/\,\mathrm{U}(1) \times \mathrm{SU}(2)_k/\,\mathrm{U}(1)$ [5] (see also subsection 3.3 for more details).

Finally, let us point out that the fact that there are no non-normalizable operators in the holographic dictionary that preserve the $\mathcal{N} = 4$ superconformal symmetry, and consequently the full worldvolume supersymmetry, ties nicely with the fact that there should not exist any such deformations of the 5+1-dimensional LST. The other class of non-normalizable operators

$$q\mu\left(J^3\Phi_{j;m} + \frac{1}{2}J^+\Phi_{j;m-1} - \frac{1}{2}J^-\Phi_{j,m+1}\right)e^{qj\phi}, \tag{2.56}$$

that preserves $\mathcal{N} = 4$, does not correspond to an LST deformation but, as we will see in more detail in section 4, moves us away from the NS5-brane horizon.

3. Deformations of $\mathrm{SU}(2)_k \times \mathbb{R}_\phi$ and NS5-branes

In this section we will analyze several configurations of NS5-branes using the holographic

operator	chiral	primary	$\mathcal{N}=2$	$\mathcal{N}=4$	spacetime susy
$\psi^+\Phi_{j;j}e^{-q(j+1)\phi}$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
$\psi^+\Phi_{j;j}e^{qj\phi}$		$\surd$			
$(\psi\Phi_j)_{j+1;m}e^{-q(j+1)\phi}$			$\surd$	$\surd$	$\surd$
$(\psi\Phi_j)_{j+1;m}e^{qj\phi}$					
$\psi_3 e^{-q\phi}$		$\surd$	$\surd$	$\surd$	$\surd$
ψ_3			$\surd$		
$(\psi\Phi_j)_m e^{-q(j+1)\phi}$					
$(\psi\Phi_j)_m e^{qj\phi}$			$\surd$	$\surd$	$\surd$
$\psi\Phi_{j;-j}e^{-q(j+1)\phi}$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
$\psi\Phi_{j;-j}e^{qj\phi}$	$\surd$		$\surd$		
$\psi_\phi e^{-q\phi}$		$\surd$	$\surd$	$\surd$	$\surd$
ψ_ϕ			$\surd$	$\surd$	$\surd$

Table 1: Properties of various classes of operators in $SU(2)_k \times \mathbb{R}_\phi$. We have not included the complex conjugates which have similar properties. Operators with unspecified j and m labels are assumed to be generic, i.e. not for the cases $j = m = 0$ and/or $m = \pm(j+1)$ when they reduce to other operators present in the table.

correspondence (2.1) and its refinement proposed in the previous section. The configurations under study will be thought of as small deformations of a stack of NS5-branes put at the point $x^6 = x^7 = x^8 = x^9 = 0$. Accordingly, the exact CFT $\mathrm{SU}(2)_k \times \mathbb{R}_\phi$ describing the latter is deformed and we will show how several physical features of the configurations of NS5-branes under study can be inferred from the analysis of the corresponding CFT deformations.

3.1 Generalities

We revisit now the holographic dictionary (2.1) and explain how it works in detail. Since there are no m and $\bar{m}$ indices at the left side, an obvious question is how these charges are determined in terms of the indices $i_1, \ldots, i_{2j+2}$ for a given LST operator. As shown in [5], this can be done by using a parametrization of the moduli space in terms of two complex variables that span the two orthogonal hyperplanes transverse to the NS5-branes:

$$A \equiv X^6 + iX^7, \qquad B \equiv X^8 + iX^9 \ . \tag{3.1}$$

Embedding the rotational $\mathrm{SO}(2)_A \times \mathrm{SO}(2)_B$ of the A and B planes in the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry of the CHS background so that $\mathrm{SO}(2)_A$ is generated by $\mathcal{J}_3 - \bar{\mathcal{J}}_3$ and $\mathrm{SO}(2)_B$ is generated by $\mathcal{J}_3 + \bar{\mathcal{J}}_3$, leads to the following charge assignments

$$m_A = \frac{1}{2}, \qquad \bar{m}_A = -\frac{1}{2}, \qquad m_B = \frac{1}{2}, \qquad \bar{m}_B = \frac{1}{2} \ . \tag{3.2}$$

Subsequently, the general recipe (2.1) takes a more precise form

$$\widetilde{\mathrm{tr}}\Big(A^x B^y (A^*)^z (B^*)^w\Big) \longleftrightarrow (\psi\bar{\psi}\Phi_j)_{j+1;m,\bar{m}} e^{-q(j+1)\phi}\,, \tag{3.3}$$

where $-(j+1) \leqslant m, \bar{m} \leqslant (j+1)$ and the positive powers x, y, z, w are related to the charges $j, m, \bar{m}$ as

$$x+y = j+1+m, \quad z+w = j+1-m, \quad y+z = j+1+\bar{m}, \quad w+x = j+1-\bar{m}\ . \tag{3.4}$$

The corresponding couplings $\lambda_{j;m,\bar{m}}$ are given by

$$\lambda_{j;m,\bar{m}} = \frac{1}{k}\widetilde{\mathrm{tr}}\Big(A^x B^y (A^*)^z (B^*)^w\Big) \tag{3.5}$$

and symmetrization is not necessary since we are interested in points in the LST moduli space where A and B are diagonal. The $1/k$ factor is introduced so that the couplings are $\mathcal{O}(1)$ in general. Furthermore, one should keep only the traceless combinations in (3.5).

The analysis of the previous section indicated that generically we should consider only the operators that are either chiral or antichiral as well as the operator with $j = m = \bar{m} = 0$. The associated couplings are

$$\lambda_{j;j+1,j+1} = \frac{1}{k}\widetilde{\mathrm{tr}}(B^{2j+2})\,, \qquad \lambda_{j;-j-1,-j-1} = \frac{1}{k}\widetilde{\mathrm{tr}}\big((B^*)^{2j+2}\big)\,, \tag{3.6}$$

$$\lambda_{j;j+1,-j-1} = \frac{1}{k}\widetilde{\mathrm{tr}}(A^{2j+2})\,, \qquad \lambda_{j;-j-1,j+1} = \frac{1}{k}\widetilde{\mathrm{tr}}\big((A^*)^{2j+2}\big)\,, \tag{3.7}$$

which are automatically traceless, and

$$\lambda_{0;0,0} = \frac{1}{k}\widetilde{\mathrm{tr}}(BB^* - AA^*)\,, \tag{3.8}$$

where the relative sign is chosen so that it is traceless.

3.2 NS5-branes on a 3-sphere

The first configuration we would like to consider is that of a continuous distribution of NS5-branes on an S^3 of radius R embedded in the transverse $\mathbb{R}^4$. This configuration is described by

$$A = R\cos\theta e^{i\phi}, \quad B = R\sin\theta e^{i\tau}\,, \tag{3.9}$$

where $\theta \in [0, \pi/2), \phi \in [0, 2\pi), \tau \in [0, 2\pi)$. We would like to approximate this distribution by a sequence of discrete ones containing k NS5-branes so that the limit $k \to \infty$ yields (3.9). Then, each of the coordinates θ, ϕ, τ on the sphere is discretized in terms of an index a, b, c as follows

$$\begin{aligned} \sin^2\theta &= \frac{a}{k_1}\,, & a &= 0, \ldots, k_1\,, \\ \phi &= \frac{2\pi b}{k_2}\,, & b &= 0, \ldots, k_2\,, \\ \tau &= \frac{2\pi c}{k_3}, & c &= 0, \ldots, k_3\ . \end{aligned} \tag{3.10}$$

We can verify that

$$\frac{1}{2\pi^2}\int \sin\theta\cos\theta d\theta d\phi d\tau = \frac{1}{k}\int dadbdc\,, \tag{3.11}$$

so that the total number of NS5-branes is $k = k_1 k_2 k_3$. Notice that we assume that the discretization is smooth and hence that k_1, k_2 and k_3 are large. The discrete distribution is described by the $k \times k$ matrices

$$A_{a,b,c} = R\sqrt{1-\frac{a}{k_1}}\, e^{\frac{2\pi i b}{k_2}}\,\mathbb{I}_c\,, \qquad B_{a,b,c} = R\sqrt{\frac{a}{k_1}}\,\mathbb{I}_b e^{\frac{2\pi i c}{k_3}}\ . \tag{3.12}$$

By definition we have $\mathrm{tr}(A) \equiv \sum_{a=0}^{k_1}\sum_{b=0}^{k_2}\sum_{c=0}^{k_3} A_{a,b,c}$ and we conclude that $\mathrm{tr}(A) = \mathrm{tr}(B) = 0$ as it should.

Before considering the chiral and antichiral operators let us see what happens with the $j = m = \bar{m} = 0$ operator. Its coupling turns out to be zero since

$$\sum_{a,b,c}\left(B_{a,b,c}B^*_{a,b,c} - A_{a,b,c}A^*_{a,b,c}\right) \sim \sum_{a=0}^{k_1}\left(\frac{2}{k_1}a - 1\right) = 0\ . \tag{3.13}$$

Hence, for this particular configuration this operator does not appear in the perturbed theory. Notice that we have used the standard trace since, as argued in [6], the multi-trace corrections are subleading when k is large and j is finite. For the same reason we will employ the usual single trace in the computation of $\lambda_{j;m,\bar{m}}$ below, since the values of j that we will consider will be large, but finite.

We proceed now with the computation of the coefficients $\lambda_{j;m,\bar{m}}$ for the cases where $m = \pm(j+1)$ and $\bar{m} = \pm(j+1)$. We have

$$\begin{aligned}\lambda_{j;m,\bar{m}} &= R^{|m-\bar{m}|+|m+\bar{m}|}\frac{1}{k}\sum_{a=0}^{k_1}\sum_{b=0}^{k_2}\sum_{c=0}^{k_3}\left(1-\frac{a}{k_1}\right)^{\frac{|m-\bar{m}|}{2}}\left(\frac{a}{k_1}\right)^{\frac{|m+\bar{m}|}{2}} e^{\frac{2\pi i b}{k_2}(m-\bar{m})}e^{\frac{2\pi i c}{k_3}(m+\bar{m})}\\ &= R^{|m-\bar{m}|+|m+\bar{m}|}\frac{k_2 k_3}{k}\sum_{a=0}^{k_1}\left(1-\frac{a}{k_1}\right)^{\frac{|m-\bar{m}|}{2}}\left(\frac{a}{k_1}\right)^{\frac{|m+\bar{m}|}{2}}\delta_{m-\bar{m},0 \bmod k_2}\delta_{m+\bar{m},0 \bmod k_3}\ .\end{aligned}$$

For large k_1, k_2, k_3 we can approximate the summation over a with an integral. We get

$$\lambda_{j;m,\bar{m}} = R^{|m-\bar{m}|+|m+\bar{m}|}B\left(1+\frac{|m-\bar{m}|}{2}, 1+\frac{|m+\bar{m}|}{2}\right)\delta_{m-\bar{m},0 \bmod k_2}\delta_{m+\bar{m},0 \bmod k_3}\,, \tag{3.14}$$

where $B(x,y)$ is the Euler beta function. Notice that if we had made the same approximation for the summations over b and c we would have found that only the coefficient with $m = \bar{m} = 0$ is non-zero, therefore missing all other possibilities.

Let us now try to understand the behavior of the coefficients $\lambda_{j;m,\bar{m}}$ for operators that are either (chiral, chiral) or (chiral, antichiral). The other two cases are related to these two by conjugation. In the first case we have $(m,\bar{m}) = (j+1, j+1)$ and the corresponding coupling is

$$\lambda_{j;j+1,j+1} = \frac{R^{2j+2}}{j+1}\delta_{2j+2,0 \bmod k_3}\ . \tag{3.15}$$

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Since $2j + 2$ goes up to k, there are $k/k_3 = k_1 k_2$ values of j which give non-vanishing $\lambda_{j;j+1,j+1}$. For the (chiral, antichiral) case, where $(m, \bar{m}) = (j+1, -j-1)$, the coupling is

$$\lambda_{j;j+1,-j-1} = \frac{R^{2j+2}}{j+1} \delta_{2j+2,0 \bmod k_2} \tag{3.16}$$

and we have $k/k_2 = k_1 k_3$ values of j that yield non-vanishing coefficients.

We can now consider the purely bosonic deformation corresponding to the (chiral, chiral) operators:

$$\sum_{j=0}^{\frac{k-2}{2}} \frac{q^2}{2} \lambda_{j;j+1,j+1} J^+ \bar{J}^+ \Phi_{j;j,j} e^{-q(j+1)\phi} \,. \tag{3.17}$$

Explicitly this reads

$$\begin{aligned} &\sum_{j=0}^{\frac{k-2}{2}} \frac{1}{k} \frac{R^{2j+2}}{j+1} J^+ \bar{J}^+ \Phi_{j;j,j} e^{-q(j+1)\phi} \; \delta_{2j+2,0 \bmod k_3} \\ &\qquad = \sum_{j=0}^{\frac{k-2}{2}} \frac{J^+ \bar{J}^+}{k(j+1)} \Phi_{j;j,j} \; e^{-q(j+1)(\phi - \sqrt{2k} \ln R)} \; \delta_{2j+2,0 \bmod k_3} \,. \end{aligned} \tag{3.18}$$

We start the analysis by noticing that the smallest value of j that contributes is of order k_3 and hence it is large. Consequently, if $\phi - \sqrt{2k} \ln R = x < 0$ the exponential is $e^{-\frac{j}{\sqrt{k}} x}$ and for j of order k_3 it creates a potential wall that does not allow penetration in the $x < 0$ region. In terms of the linear dilaton coordinate this wall is located at $\phi_0 = \sqrt{2k} \ln R$. The region $x > 0 \Leftrightarrow \phi > \phi_0$ has a potential that goes very rapidly to zero as $k \to \infty$. Hence, when $k \to \infty$ we obtain the original $\mathrm{SU}(2) \times \mathbb{R}_\phi$ theory, but with a truncated dilaton direction $\phi > \phi_0$. In terms of the usual radial coordinate the wall is located exactly at the radius of the 3-sphere.

What we have just described is a stringy way of creating an impenetrable domain for all modes in the theory. This should be compared to a purely gravitational approach in which one postulates, without providing a microscopic origin, that the region $r < R$ is not part of the space and δ-function source terms are added so that the equations of motion are satisfied at $r = R$.

This analysis is in perfect agreement with the application of the Gauss law for a configuration of NS5-branes spread on a 3-sphere. In the limit of large k the $\mathrm{SO}(4)$ symmetry is restored and according to the Gauss law, outside of the 3-sphere we should obtain the same solution as that of a point-like configuration of NS5-branes, i.e. $\mathrm{SU}(2)_k \times \mathbb{R}_\phi$, but for the fact that the dilaton direction is truncated to $\phi > \phi_0$. Instead, inside the sphere we have no sources and the solution should be the free CFT corresponding to four free bosons parametrizing $\mathbb{R}^4$. However, this effect cannot be seen explicitly in our analysis since the perturbation blows up and cannot be considered as a small deformation of the original CFT.

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3.3 NS5-branes on a circle: redux

We consider now a configuration of k NS5-branes arranged symmetrically on a polygon inscribed in a circle of radius R in the B plane. This is a configuration that has been discussed extensively in the literature since it admits an exact CFT description. It is described by the $k \times k$ matrices

$$A_a = 0, \quad B_a = Re^{2\pi i a/k}\,, \tag{3.19}$$

so that the only non-zero couplings are $\lambda_{0;0,0} = R^2$ and $\lambda_{\frac{k-2}{2};\frac{k}{2},\frac{k}{2}}\lambda_{\frac{k-2}{2};-\frac{k}{2},-\frac{k}{2}}R^k$. Notice that $\mathrm{tr}(B^l) = 0$ when $l < k$ and hence there is no difference between the usual trace and the one containing multi-traces.

The operator corresponding to the second coupling behaves exactly as the operators discussed in the previous subsection, i.e. it rapidly vanishes when we are probing the region outside the ring. Hence, the theory is modified only by the presence of the operator corresponding to $\lambda_{0;0,0} = R^2$, which is $\psi_3\bar{\psi}_3 e^{-q\phi}$ and whose bosonic part reads $J_3\bar{J}_3 e^{-q\phi}$. This operator drives the deformation

$$\mathrm{SU}(2)_k \times \mathbb{R}_\phi = \frac{\mathrm{SU}(2)_k}{\mathrm{U}(1)} \times \mathrm{U}(1) \times \mathbb{R}_\phi \longrightarrow \frac{\mathrm{SU}(2)_k}{\mathrm{U}(1)} \times \frac{\mathrm{SL}(2,\mathbb{R})_k}{\mathrm{U}(1)}\,, \tag{3.20}$$

where a T-duality has been also been performed in the first step. The latter model is indeed well-known to provide the exact CFT description of a circular configuration of NS5-branes in the near-horizon limit [2].

We may verify explicitly that statement from the expressions for the corresponding background [2]

$$\begin{aligned} ds^2 &= k\left[d\rho^2 + d\theta^2 + \frac{1}{\Sigma}(\tanh^2\rho\ d\tau^2 + \tan^2\theta\ d\psi^2)\right], \\ B_{\tau\psi} &= \frac{k}{\Sigma}\,, \\ e^{-2\Phi} &= \Sigma\cosh^2\rho\cos^2\theta\,, \end{aligned} \tag{3.21}$$

where the dilaton is included for completeness and

$$\Sigma = \tanh^2\rho\tan^2\theta + 1\ . \tag{3.22}$$

Indeed, expanding (3.21) for large ρ which, effectively, is equivalent to a circle of small size, we get that

$$\begin{aligned} ds^2 &= ds^2_{(0)} + 4e^{-2\rho}(\sin^4\theta\ d\psi^2 - \cos^4\theta\ d\tau^2) + \mathcal{O}\left(e^{-4\rho}\right)\,, \\ B_{\tau\psi} &= B^{(0)}_{\tau\psi} + 4e^{-2\rho}\sin^2\theta\cos^2\theta + \mathcal{O}\left(e^{-4\rho}\right)\,, \end{aligned} \tag{3.23}$$

where the fields indexed with a zero correspond to the SU(2) WZW unperturbed case. It is now straightforward to verify that the perturbation is just $-J_3\bar{J}_3 e^{-q\phi}$, where $\phi = \sqrt{2k}\ \rho$ and

$$J_3 = 2(\cos^2\theta\partial\tau + \sin^2\theta\partial\psi)\,, \qquad \bar{J}_3 = 2(\cos^2\theta\bar{\partial}\tau - \sin^2\theta\bar{\partial}\psi)\,, \tag{3.24}$$

are the chirally and antichirally conserved Cartan currents.

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3.4 NS5-branes on orthogonal circles

Another interesting configuration is that of k NS5-branes put on two circles of the same radius R on the two planes A and B. This is described by the $k' \times k'$ matrices

$$A_a = Re^{2\pi i a/k'}, \quad B_a = Re^{2\pi i a/k'}, \tag{3.25}$$

with $k' = k/2$. In this case the only non-zero couplings are $\lambda_{\frac{k'-2}{2};\pm\frac{k'}{2},\pm\frac{k'}{2}} = \frac{1}{2}R^{k'}$. As in the previous case of NS5-branes put on a single circle, there is no difference between the usual trace and the tilde one. For large $k' = k/2$ the corresponding operators vanish rapidly in the region outside of the two rings and since there are no other non-vanishing operators (unlike the case of one ring where $\lambda_{0;0,0} \neq 0$) the situation resembles that of the 3-sphere.

The previous remark leads to a puzzle, since the solutions corresponding to the 3-sphere and the two circles are obviously different even in the $k \to \infty$ limit. One can understand why the two configurations behave similarly by examining the multipole expansion of the corresponding harmonic functions, in conjunction also with a similar expansion for the case of one circle. The reason is that on physical grounds the first non-vanishing multipole moment triggers the leading perturbation of the original CFT. Since the 3-sphere configuration behave as a point-like charge, all of its multipole moments vanish by definition. The dipole moments $p^i = \int d^4x\rho(x)x^i$, where $i = 6, 7, 8, 9$ is a vector index in the 4-dimensional transverse $\mathbb{R}^4$ and $\rho(x^i)$ is the density of NS5-branes, are zero for both the case of one and two circles. This is easy to check using the normalized densities

$$\rho_{1-\text{circ.}}(x) = \frac{1}{\pi}\delta\left(R^2 - (x^8)^2 - (x^9)^2\right)\delta(x^6)\delta(x^7) \tag{3.26}$$

and

$$\rho_{2-\text{circ.}}(x) = \frac{1}{2\pi}\Big[\delta\left(R^2 - (x^8)^2 - (x^9)^2\right)\delta(x^6)\delta(x^7) + \delta\left(R^2 - (x^6)^2 - (x^7)^2\right)\delta(x^8)\delta(x^9)\Big]\,. \tag{3.27}$$

Now, one can further check that the quadrupole moments

$$Q^{ij} = \int d^4x\rho(x)\left(x^ix^j - \frac{1}{4}\delta^{ij}x^2\right), \tag{3.28}$$

vanish for the two circles, rendering the solution similar to that of the 3-sphere up to this order. Instead, the single circle behaves differently since it has non-vanishing quadrupole moments, for instance $Q_{66} = Q_{77} = -\frac{1}{4}R^2$. More generally, the quadrupole moments vanish for every distribution that is identical on the A and B planes and which has no dependance on the angular coordinate of the plane.

3.5 NS5-branes on a line and symmetry considerations

A final configuration we would like to consider is that of NS5-branes put on a line, for instance in the B plane and along the x^8 direction. In that case we have $A = 0$ and $B = B^*$ and all couplings $\lambda_{j;j+1,j+1}$, their conjugates as well as $\lambda_{0;0,0}$ are generically non-zero. Notice that our discussion here is independent of the actual distribution on the line.

The configuration we consider is invariant under an SO(3) group of transverse symmetries and a natural question is how this symmetry manifests itself in the CFT deformations.

Before tackling this problem, let us start with a generic configuration of NS5-branes in the transverse $\mathbb{R}^4$ where both A and B are non-zero. The various cases will be discussed in reference to figure 1 below which summarizes and depicts them geometrically. Then, generically, all the couplings $\lambda_{j;\pm(j+1),\pm(j+1)}$ and $\lambda_{0;0,0}$ are non-zero and the $\mathrm{SO}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry is completely broken. Instead, an arbitrary deformation on a single plane should preserve the SO(2) symmetry associated with rotations in the plane orthogonal to the first one. For instance, spreading the branes in the B plane triggers the (chiral, chiral) and (antichiral, antichiral) operators corresponding to the couplings $\lambda_{j;j+1,j+1}$ and $\lambda_{j;-j-1,-j-1}$ respectively, as well as $\lambda_{0;0,0}$. The purely bosonic pieces of the associated deformations are

$$J^+ \bar{J}^+ \Phi_{j;j,j} e^{-q(j+1)\phi} , \qquad J^- \bar{J}^- \Phi_{j;-j,-j} e^{-q(j+1)\phi} , \qquad J^3 \bar{J}^3 e^{-q(j+1)\phi} . \tag{3.29}$$

All these operators are invariant[4] under the generator $J^3 - \bar{J}^3$ of $\mathrm{SO}(2)_A$.[5] Notice that, had we spread the branes in the A plane, the purely bosonic deformations would have been proportional to

$$J^+ \bar{J}^- \Phi_{j;j,-j} e^{-q(j+1)\phi} , \qquad J^- \bar{J}^+ \Phi_{j;-j,j} e^{-q(j+1)\phi} , \qquad J^3 \bar{J}^3 e^{-q(j+1)\phi} . \tag{3.30}$$

These are now invariant under $J^3 + \bar{J}^3$, i.e. the generator of $\mathrm{SO}(2)_B$, as they should.

If we put now the NS5-branes on a regular polygon on the B plane, which approaches a smooth circle in the $k \to \infty$, the symmetry we expect is $\mathrm{SO}(2)_A \times \mathbb{Z}_k$, while its continuous limit should be $\mathrm{SO}(2)_A \times \mathrm{SO}(2)_B$. The commutator of the $\mathrm{SO}(2)_B$ generator $J^3 + \bar{J}^3$ with the $j = m = \bar{m} = 0$ deforming operator is zero, while with the only other operator that is turned on, i.e. that with $j = (k-2)/2$, it yields

$$k J^+ \bar{J}^+ \Phi_{\frac{k-2}{2};\frac{k-2}{2},\frac{k-2}{2}} e^{-q(j+1)\phi} \tag{3.31}$$

and similarly for its conjugate. In other words the deformation has charge k and therefore is invariant under rotations by $2\pi/k$, i.e. there is indeed a discrete $\mathbb{Z}_k$ symmetry. More generally, operators with a given j have an invariance under $\mathbb{Z}_{2j+2}$ since they correspond to the deformations of [7] that have indeed such a geometric symmetry.

Before discussing the case of the bar, let us point out that there are two relevant SO(3) subgroups of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, the latter being generated by $J^\pm, J^3$ and $\bar{J}^\pm, \bar{J}^3$. The first, which we call $\mathrm{SO}(3)_B$ is generated by

$$\mathrm{SO}(3)_B : \quad J^3 + \bar{J}^3, \quad J^+ + \epsilon \bar{J}^+, \quad J^- + \epsilon^{-1} \bar{J}^- \tag{3.32}$$

with ϵ an arbitrary complex number and one of its quadratic invariants is

$$\epsilon^{-1} J^+ \bar{J}^- + \epsilon J^- \bar{J}^+ + 2 J^3 \bar{J}^3 . \tag{3.33}$$

[4]As usual, when we refer to the action of the currents we mean the action of their zero-modes and hence what we should check is the simple pole in their OPE with the operator under consideration.

[5]We consider J_3 instead of $\mathcal{J}_3$ since we focus on the purely bosonic deformations but obviously the same argument extends to the fermion bilinear pieces.

This invariant is part of the full Casimir constructed out of the generators (3.32); the latter is actually the sum of (3.33) and of the usual Casimirs made out of J^3, $J^\pm$ and $\bar{J}^3$, $\bar{J}^\pm$, for $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$, respectively.

The other relevant SO(3) subgroup of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, which will be denoted by $\mathrm{SO}(3)_A$, is generated by

$$\mathrm{SO}(3)_A : \quad J^3 - \bar{J}^3, \quad J^+ + \epsilon \bar{J}^-, \quad J^- + \epsilon^{-1} \bar{J}^+ \tag{3.34}$$

and its invariant, analogous to (3.33), is

$$\epsilon^{-1} J^+ \bar{J}^+ + \epsilon J^- \bar{J}^- - 2J^3 \bar{J}^3 \ . \tag{3.35}$$

Notice that demanding reality of these invariants requires ϵ to be a phase. The corresponding $\mathrm{SO}(2)_{A/B}$ subgroups are generated by $J^3 \pm \bar{J}^3$ according to the convention we established at the beginning of this section.

Let us consider now a configuration of NS5-branes arbitrarily spread along a line in the B plane passing by the center. Such a line is described by $B = e^{i\varphi} B^*$ where φ is twice the angle it makes with the x^8 axis. For $j = 0$ the standard trace and the tilde one are identical and we find that the couplings $\lambda_{0;1,1}$, $\lambda_{0;-1,-1}$ and $\lambda_{0;0,0}$ are related as

$$\lambda_{0;-1,-1} = e^{-i\varphi} \lambda_{0;0,0} \,, \qquad \lambda_{0;1,1} = e^{i\varphi} \lambda_{0;0,0} \ . \tag{3.36}$$

Subsequently, the purely bosonic part of the corresponding deformation is given by the operator

$$\lambda_{0;0,0} (e^{i\varphi} J^+ \bar{J}^+ + e^{-i\varphi} J^- \bar{J}^- + 2J^3 \bar{J}^3) e^{-q\phi} \,, \tag{3.37}$$

where the factor of 2 in front of the last term appears because in (2.22) we are instructed to add the complex conjugate of every term. According to the discussion above, this operator is indeed invariant under $\mathrm{SO}(3)_A$ for $\epsilon = -e^{-i\varphi}$.

A similar situation would have arisen if we had put the NS5-branes on a bar in the A plane, with $\mathrm{SO}(3)_B$ being now the relevant symmetry group. Notice also that this argument works in reverse. If we have a configuration with $\mathrm{SO}(3)_A$ symmetry (and such that $\mathrm{tr}(B^2) \neq 0$), the unique invariant that depends on both holomorphic and antiholomorphic currents is $\epsilon^{-1} J^+ \bar{J}^+ + \epsilon J^- \bar{J}^- - 2J^3 \bar{J}^3$ and therefore it dictates the following relations between the couplings: $\lambda_{0;1,1} = -\epsilon^{-1} \lambda_{0;0,0}$ and $\lambda_{0;-1,-1} = -\epsilon \lambda_{0;0,0}$. In other words, $\mathrm{tr}(B^2) = -\epsilon^{-1} \mathrm{tr}(BB^*)$ and $\mathrm{tr}\big((B^*)^2\big) = -\epsilon \mathrm{tr}(BB^*)$, therefore implying that ϵ is a phase and furthermore that[6] $B = -\epsilon^{-1} B^*$. Hence, only a configuration of NS5-branes along a line can have $\mathrm{SO}(3)_{A/B}$ symmetry.

Since operators with $j > 0$ are also turned on, corresponding to the couplings $\widetilde{\mathrm{tr}}(B^{2j+2})$ which are generically non-vanishing, one should also check that the associated

[6]Proof: we would like to show that if $\mathrm{tr}(B^2) = e^{i\varphi} \mathrm{tr}(BB^*)$ then $B = e^{i\varphi} B^*$. Define $\Gamma = e^{-i\varphi/2} B$ so that $\mathrm{tr}(\Gamma^2) = \mathrm{tr}(\Gamma\Gamma^*)$. B and hence Γ are diagonal matrices and let $r_i e^{i\theta_i}$ by the elements of the latter. We can re-write the trace condition on Γ as $\sum_i (r^i)^2 (e^{2i\theta_i} - 1) = 0$. The real part of this equation gives $-2\sum_i (r^i)^2 \sin^2\theta_i = 0$, therefore implying that θ_i is an integer multiple of π. Hence $\Gamma = \Gamma^*$ and consequently $B = e^{i\varphi} B^*$.

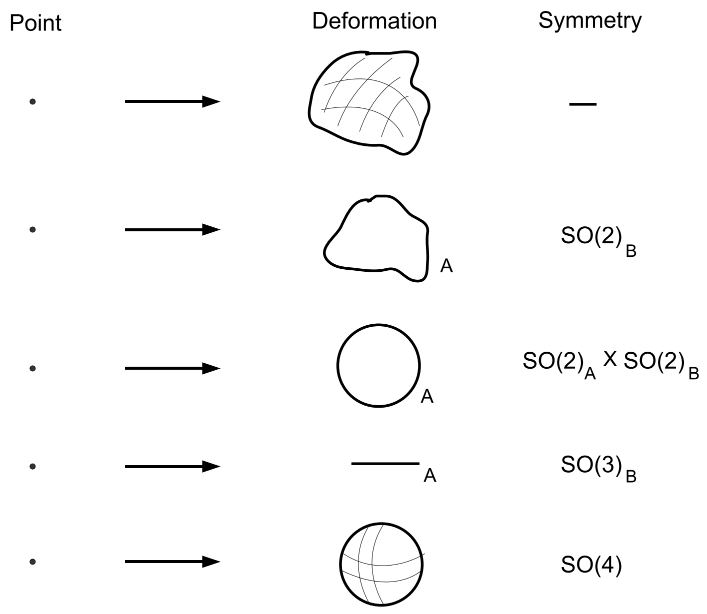

Figure 1: Various NS5-branes configurations and their symmetries in the continuous ($k \to \infty$) limit: (i) generic distribution in $\mathbb{R}^4$ with no symmetry, (ii) generic planar deformation with SO(2) symmetry, (iii) circle with SO(2) $\times$ SO(2) symmetry, (iv) bar with SO(3) symmetry, (v) 3-sphere with SO(4) symmetry.

deformations are $\mathrm{SO}(3)_A$ invariant. The purely bosonic piece of the deformation corresponding to $\widetilde{\mathrm{tr}}(B^{2j+2})$ is $J^+\bar{J}^+\Phi_{j;j,j}e^{-q(j+1)\phi}$ and we have to include also its conjugate $J^-\bar{J}^-\Phi_{j;-j,-j}e^{-q(j+1)\phi}$. These operators are separately invariant under $J^3 - \bar{J}^3$ as we have already pointed out. However, they are not invariant under the other two generators of $\mathrm{SO}(3)_A$ since they correspond to deformations of higher order in the deforming parameter and we expect that the actual generators of the SO(3) symmetry are also corrected beyond the leading order. It would quite interesting to uncover the corrected form of the symmetry generators beyond the leading order. However, this task is tantamount to constructing the CFT underlying this configuration and therefore it should be quite non-trivial.

We have summarized the symmetries of various NS5-brane configurations in figure 1.

4. "Non-holographic" CHS deformations

A four-dimensional metric conformal to a hyperkähler one supports generically $\mathcal{N} = (4,1)$ world-sheet supersymmetry with torsion, provided that the conformal factor is a harmonic function of the hyperperkähler metric. The background fields are

$$ds^2 = H ds^2_{\mathrm{HK}}\,, \qquad \mathcal{H}_{ijk} = \epsilon_{ijk}{}^{l}\partial_l H\,, \qquad e^{2\Phi} = H\,, \tag{4.1}$$

with $\nabla^2_{\mathrm{HK}} H = 0$ and where the indices are raised with the hyperkähler metric. We present below one such example that was worked out as a gravity solution in [14, 15] and we formulate it in the language of $\mathrm{SU}(2)_k \times \mathbb{R}_\phi$ deformations. The corresponding operator is not chiral (or antichiral) primary and hence, according to our proposal in subsection 2.5, does not correspond to a geometric NS5-brane deformation.

Subsequently we examine the case where the hyperperkähler space is provided by $\mathbb{R}^4$ and present an interesting application concerning the leading correction of $\mathrm{SU}(2)_k \times \mathbb{R}_\phi$ towards the full NS5-brane solution. The correction is triggered by one of the non-normalizable operators found in section 2 which preserve $\mathcal{N} = 4$ supersymmetry but do not have holographic interpretation.

4.1 Conformal Eguchi-Hanson metric

This example is provided by taking the Eguchi-Hanson as the hyperkähler metric [16]

$$\frac{ds^2_{\mathrm{EH}}}{2k} = \frac{r^4}{r^4 - a^4}\, dr^2 + r^2(\sigma_1^2 + \sigma_2^2) + \frac{r^4 - a^4}{r^2}\, \sigma_3^2 \,, \tag{4.2}$$

where σ_a, $a = 1, 2, 3$ are the Maurer-Cartan SU(2) right-invariant 1-forms, normalized as

$$d\sigma_a = \epsilon_{abc}\sigma_b \wedge \sigma_c \ . \tag{4.3}$$

We have also introduced an overall scale $2k$ for later convenience. Assuming that the conformal factor H depends only on the radial variable r, we easily establish that

$$H = \frac{A}{2a^2} \ln\left(\frac{r^2 + a^2}{r^2 - a^2}\right) + B \,, \tag{4.4}$$

where A and B are integration constants. In addition, we note that the antisymmetric NS 3-form field strength is independent of a and is given by

$$\mathcal{H} = 2\sigma_1 \wedge \sigma_2 \wedge \sigma_3 \ . \tag{4.5}$$

We select $A = 1$ and $B = 0$ so that for small a, or equivalently large r, the conformal factor becomes

$$H = \frac{1}{r^2}\left[1 + \frac{a^4}{3r^4} + \mathcal{O}\left(\frac{a^8}{r^8}\right)\right] \ . \tag{4.6}$$

In this limit the background corresponds to a deformation of $\mathrm{SU}(2)_k \times \mathbb{R}_\phi$, with the leading worldsheet correction being proportional to

$$a^4 \left(J^1\bar{K}^1 + J^2\bar{K}^2 - 2J^3\bar{K}^3\right) e^{-2q\phi} \,, \tag{4.7}$$

where we have changed variables as

$$r = e^{q\phi/2} \qquad \Longrightarrow \qquad \Phi = -\frac{q}{2}\phi \tag{4.8}$$

in order to make the dilaton linear to leading order. We have also performed a ϕ-dependent reparametrization that keeps the coefficient of $d\phi^2$ equal to one.

The currents that appear in the deformation (4.7) are given by

$$J^a = -i\mathrm{tr}(\partial g g^{-1}\tau^a) \,, \quad \bar{K}^a = -i\mathrm{tr}(\bar{\partial} g g^{-1}\tau^a) \,, \qquad a = 1, 2, 3 \,, \tag{4.9}$$

with τ^a being the Pauli matrices. Note that, whereas the current J^a is the chirally conserved current of SU(2), obeying $\bar\partial J^a = 0$, the current $\bar K^a$ is not its antiholomorphic counterpart, i.e. $\bar\partial K^a \neq 0$. However, one can write

$$\bar K^a = C^a{}_b \bar J^b\,, \qquad C_{ab} = \frac{1}{2}\mathrm{tr}(\tau_a g \tau_b g^{-1})\,, \tag{4.10}$$

where the currents

$$\bar J^a = -i\mathrm{tr}(g^{-1}\bar\partial g \tau^a)\,, \tag{4.11}$$

are indeed antiholomorphic obeying $\partial \bar J^a = 0$. The matrix C_{ab} is in the adjoint representation.

In order to make contact with the expressions for the SU(2) primaries of spin 1 consider the group element in the spin 1/2-representation parametrized as

$$g = \begin{pmatrix} g_{++} & g_{+-} \\ g_{-+} & g_{--} \end{pmatrix}\,, \tag{4.12}$$

from which we compute that[7]

$$\begin{aligned} C^{\pm\pm} &= -2g^2_{\pm\mp}\,, \qquad & C^{\pm\mp} &= 2g^2_{\pm\pm}\,, \\ C^{3\pm} &= \pm 2 g_{\mp\mp} g_{\pm\mp}\,, \qquad & C^{\pm 3} &= \mp 2 g_{\pm\pm} g_{\pm\mp}\,, \qquad & C^{33} &= g_{+-}g_{-+} + g_{++}g_{--}\,. \end{aligned} \tag{4.13}$$

On the other hand let us recall the transformation rules

$$\delta_\pm g_{\mp a} = g_{\pm a}\,, \qquad \delta_3 g_{\pm a} = \pm\frac{1}{2} g_{\pm a}\,, \qquad a = \pm\,, \tag{4.14}$$

for the left SU(2) action on the group element, a similar one for the right action and the fact that for spin j state with $m = \bar m = j$ is given by

$$\Phi_{j;j,j} = g^{2j}_{++}\,. \tag{4.15}$$

The other members of the representation are then obtained by acting with the above transfromation rules. It is important to normalize the states generated in this way in a fashion compatible with the OPEs (2.5). This is done if

$$\Phi_{j;m\pm1,\bar m} = \frac{1}{j\mp m}\,\delta_\pm \Phi_{j;m,\bar m}\,, \tag{4.16}$$

for the left as well as for the right SU(2) transformations.

Using this method we easily find

$$\begin{aligned} \Phi_{1;\pm1,\pm1} &= g^2_{\pm\pm}\,, \qquad & \Phi_{1;\pm1,0} &= g_{\pm\pm}g_{\pm\mp}\,, \\ \Phi_{1;\pm1,\mp1} &= g^2_{\pm\mp}\,, \qquad & \Phi_{1;0,\pm1} &= g_{\pm\pm}g_{\mp\pm}\,, \qquad & \Phi_{1;0,0} &= \frac{1}{2}(g_{++}g_{--} + g_{+-}g_{-+})\,. \end{aligned} \tag{4.17}$$

[7] We use the basis $\sigma^\pm = \sigma_1 \mp i\sigma_2$, σ_3, where the indices are raised and lowered with the metric $\eta_{33} = 2\eta_{+-} = 2\eta_{+-} = 1$.

Comparing now these expressions with (4.13) we can finally write the elements of the matrix C^{ab} in terms of the primaries $\Phi_{j;m,\bar{m}}$ for $j=1$:

$$\begin{aligned} C^{\pm\pm} &= -2\Phi_{1;\pm1,\mp1}\,, & C^{\pm\mp} &= 2\Phi_{1;\pm1,\pm1}\,, \\ C^{3\pm} &= \pm2\Phi_{1;0,\mp1}\,, & C^{\pm3} &= \mp2\Phi_{1;\pm1,0}\,, \qquad C^{33} = 2\Phi_{1;0,0}\,. \end{aligned} \tag{4.18}$$

The currents $\bar{K}^a$ can be written in terms of the antiholomorphic currents $\bar{J}^a$ as

$$\begin{aligned} \bar{K}^{\pm} &= \frac{1}{2}C^{\pm+}\bar{J}^- + \frac{1}{2}C^{\pm-}\bar{J}^+ + C^{\pm3}\bar{J}^3\,, \\ \bar{K}^{3} &= \frac{1}{2}C^{3+}\bar{J}^- + \frac{1}{2}C^{3-}\bar{J}^+ + C^{33}\bar{J}^3\,, \end{aligned} \tag{4.19}$$

and by using the relations (4.18) we obtain the explicit representation

$$\begin{aligned} \bar{K}^{+} &= -\Phi_{1;1,-1}\bar{J}^- + \Phi_{1;1,1}\bar{J}^+ - 2\Phi_{1;1,0}\bar{J}^3\,, \\ \bar{K}^{-} &= \Phi_{1;-1,-1}\bar{J}^- - \Phi_{1;-1,1}\bar{J}^+ + 2\Phi_{1;-1,0}\bar{J}^3\,, \\ \bar{K}^{3} &= \Phi_{1;0,-1}\bar{J}^- - \Phi_{1;0,1}\bar{J}^+ + 2\Phi_{1;0,0}\bar{J}^3\,. \end{aligned} \tag{4.20}$$

We can now rewrite the deformation (4.7) as

$$a^4\left(\frac{1}{2}J^+\bar{K}^- + \frac{1}{2}J^-\bar{K}^+ - 2J^3\bar{K}^3\right)e^{-2q\phi}\,, \tag{4.21}$$

and then, by using (4.20), we can finally express the deformation as

$$\begin{aligned} a^4\Bigg[&\left(-\frac{1}{2}\Phi_{1;1,-1}J^- + \frac{1}{2}\Phi_{1;-1,-1}J^+ - 2\Phi_{1;0,-1}J^3\right)\bar{J}^- \\ &+\left(\frac{1}{2}\Phi_{1;1,1}J^- - \frac{1}{2}\Phi_{1;-1,1}J^+ + 2\Phi_{1;0,1}J^3\right)\bar{J}^+ \\ &+\left(-\Phi_{1;1,0}J^- + \Phi_{1;-1,0}J^+ - 4\Phi_{1;0,0}J^3\right)\bar{J}^3\Bigg]e^{-2q\phi}\,. \end{aligned} \tag{4.22}$$

Comparing the first line above with (2.45) we find that they match for $j=1$ and $m=0$. From (2.46) we get that $\mu_\pm = \pm1/\sqrt{2}$ and $\mu=-2$. Then (2.45) reproduces the first line in the expression above. Similarly, the other two lines in the above expressions match precisely, up to a multiplicative factor, with (2.45) for the same values of $j, m, \mu_\pm$ and μ. Therefore, the results of the first section imply that the holomorphic pieces of the deformation (4.7) preserve $\mathcal{N}=4$ supersymmetry. Instead, the antiholomorphic ones, given by $\bar{K}^\pm$ and $\bar{K}^3$ in (4.20), preserve neither $\mathcal{N}=2$ nor $\mathcal{N}=4$ supersymmetry. Hence, the total supersymmetry of the background (4.1) is $\mathcal{N}=(4,1)$, in agreement with the analysis of [14, 15].

We note that, by considering more general solutions than (4.4) for the conformal factor H, we can construct perturbations corresponding to operators with spin $j>1$.

There is also an interesting interpretation of the background (4.1) in terms of NS5-branes. It is easy to see that the backreaction of a configuration of NS5-branes put transversely on a hyperkähler space corresponds exactly to the fields in (4.1). In the particular case where the transverse hyperkähler space is the orbifold limit of an ALE space, such systems were studied in the context of LST holography in [17]. The fact that worldsheet supersymmetry is reduced from $\mathcal{N} = (4,4)$ to $\mathcal{N} = (4,1)$ can be understood from this point of view as follows. Consider type IIB theory where the worldvolume of a set of parallel NS5-branes, with transverse $\mathbb{R}^4$, supports $\mathcal{N}_6 = (1,1)$ supersymmetry (by $\mathcal{N}_6$ we mean six-dimensional supersymmetries). If instead of the NS5-brane geometry we had a hyperkähler space, the supersymmetry in the remaining six-dimensional Minkowski space would be $\mathcal{N}_6 = (2,0)$. Hence, superimposing the NS5-branes with the hyperkähler space, so that they share a common six-dimensional Minkowski spacetime, leads to $\mathcal{N}_6 = (1,0)$ supersymmetry. The latter necessitates the presence of $\mathcal{N} = (4,1)$ in the worldsheet theory, in accordance with the previous discussion. Had we considered type IIA string theory, the NS5-brane supersymmetry would have been $\mathcal{N}_6 = (2,0)$ whereas the hyperkähler space would have preserved $\mathcal{N}_6 = (1,1)$, therefore leading to the same result.

4.2 Conformal $\mathbb{R}^4$: beyond the near-horizon

In the case where the hyperkähler space in (4.1) is $\mathbb{R}^4$, we deal with a background that corresponds to a configuration of NS5-branes whose distribution is specified by the harmonic function H. Such backgrounds generically exhibit $\mathcal{N} = (4,4)$ superconformal invariance.

A very simple but quite interesting application of that construction is the following. Recall that the near-horizon geometry $\mathrm{SU}(2)_k \times \mathbb{R}_\phi$ arises from the original solution corresponding to a point-like configuration of NS5-branes by going to the near-horizon limit $r \to \infty$, which is tantamount to dropping the "1" from the harmonic function. We would like to consider the restoration of the full solution, i.e. creating a constant term in the harmonic function, as a deformation of the CHS background. This deformation reads

$$\left(J^1\bar{K}^1 + J^2\bar{K}^2 + J^3\bar{K}^3\right)e^{q\phi} = \left(\frac{1}{2}J^+\bar{K}^- + \frac{1}{2}J^-\bar{K}^+ + J^3\bar{K}^3\right)e^{q\phi}\,, \tag{4.23}$$

where we performed the usual coordinate redefinition $r = e^{q\phi/2}$.

Using the explicit form (4.20) of the currents $\bar{K}^3, \bar{K}^\pm$ we can re-write this deformation as

$$\begin{aligned}\Bigg[&\left(-\frac{1}{2}\Phi_{1;1,-1}J^- + \frac{1}{2}\Phi_{1;-1,-1}J^+ + \Phi_{1;0,-1}J^3\right)\bar{J}^- \\ &+ \left(\frac{1}{2}\Phi_{1;1,1}J^- - \frac{1}{2}\Phi_{1;-1,1}J^+ - \Phi_{1;0,1}J^3\right)\bar{J}^+ \\ &+ \left(-\Phi_{1;1,0}J^- + \Phi_{1;-1,0}J^+ + 2\Phi_{1;0,0}J^3\right)\bar{J}^3\Bigg]e^{q\phi}\,.\end{aligned} \tag{4.24}$$

Another form, equal to the previous, is obtained by using the fact that the SU(2) part in (4.23) is the quadratic Casimir and hence we can replace the right-invariant currents $J^a, \bar{K}^a$ with left-invariant ones $\bar{J}^a, K^a$:

$$\left(\frac{1}{2}J^+\bar{K}^- + \frac{1}{2}J^-\bar{K}^+ + J^3\bar{K}^3\right)e^{q\phi} = \left(\frac{1}{2}K^+\bar{J}^- + \frac{1}{2}K^-\bar{J}^+ + K^3\bar{J}^3\right)e^{q\phi}\,. \tag{4.25}$$

Hence, the deformation can be written in an equivalent way as

$$\begin{aligned}\Bigg[& \left(-\frac{1}{2}\Phi_{1;-1,1}\bar{J}^- + \frac{1}{2}\Phi_{1;-1,-1}\bar{J}^+ + \Phi_{1;-1;0}\bar{J}^3\right) J^- \\ & + \left(\frac{1}{2}\Phi_{1;1,1}\bar{J}^- - \frac{1}{2}\Phi_{1;1,-1}\bar{J}^+ - \Phi_{1;1,0}\bar{J}^3\right) J^+ \\ & + \left(-\Phi_{1;0,1}\bar{J}^- + \Phi_{1;0,-1}\bar{J}^+ + 2\Phi_{1;0,0}\bar{J}^3\right) J^3 \Bigg] e^{q\phi} \ .\end{aligned} \tag{4.26}$$

In the first expression (4.24) we see that the $\mathcal{N} = 4$ preserving non-normalizable operator of section 2 appears for the values of $j = 1, m = 0$. In the second expression (4.26) the same operator appears, this time in in the antiholomorphic sector, with $j = 1, \bar{m} = 0$. Therefore the deformation preserves $\mathcal{N} = (4, 4)$ superconformal invariance as well as spacetime supersymmetry, in accordance with the fact that the full NS5-brane solution exhibits that amount of supersymmetry. Notice that this deformation was expected to be non-normalizable since the near-horizon CHS background and the full NS5-brane solutions have different asymptotic geometries. For other values of $j, m, \bar{m}$ they should correspond to more general solutions for the harmonic function H.

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